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METHOD FOR COMPUTING THE DEFORMATION OF ELASTIC COMPONENTS IN LOAD-BEARING STRUCTURES

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Keywords: elastic plate deformation, compressive loading, transverse deflection, flexural rigidity, buckling stability, clamped plate boundary conditions, analytical solution, transcendental equation, thin plate theory.
Abstract. This work presents analytical solutions for the deflection of an elastic plate clamped at both ends and subjected to compressive loading. Closed-form expressions for deformation are derived, demonstrating that the maximum transverse deflection depends on the plate's width, thickness, length, compressive force magnitude, and flexural rigidity. A numerical example based on actual experimental data is provided, and the computed results are compared with measured values. The comparison confirms that the proposed formula yields high accuracy when applied within the allowable parameter range.

МЕТОД РАСЧЕТА ДЕФОРМАЦИИ УПРУГИХ ЭЛЕМЕНТОВ НЕСУЩИХ КОНСТРУКЦИЙ

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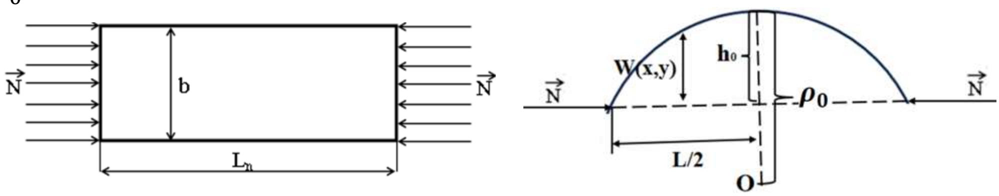
Ключевые слова: деформация упругой пластины, сжимающая нагрузка, поперечный прогиб, изгибная жесткость, устойчивость к изгибу, граничные условия жесткого защемления пластины, аналитическое решение, трансцендентное уравнение, теория тонких пластин.
Аннотация. В работе представлено аналитическое решение для определения прогиба упругой пластины, жестко закрепленной с обоих концов и подвергающейся сжимающей нагрузке. Выведены формулы для вычисления деформации, показывающие, что максимальный поперечный прогиб зависит от ширины, толщины и длины пластины, а также от величины сжимающей силы и изгибной жесткости. Приведен численный пример, основанный на реальных экспериментальных данных, а расчетные результаты сопоставлены с измеренными значениями. Сравнение подтверждает, что предложенная формула обеспечивает высокую точность при использовании в допустимом диапазоне параметров.

Introduction

Strength, rigidity, and stability are key design criteria for engineering structures. Slender members like rods and thin plates lose stability under critical loads, leading to buckling. Early research dates back to S.P. Timoshenko [1, 2], but accurate computational formulas remain a challenge. This study derives an analytical relation between deflection and geometric/material properties for a longitudinally compressed plate with fixed ends.

Mathematic model

Consider a thin rectangular plate or a flexible strip of length L_n, width b ($b < L_n$), and thickness δ, $\delta \ll b$. The plate is compressed along its longitudinal axis by a force N, uniformly distributed over the end cross-section (see Fig. 1). The objective is to determine the force N, the longitudinal deflection amplitude h_0, and the corresponding radius of curvature ρ_0, at which the transverse deflection Δh_0 attains its maximum value.



a b

Fig. 1. Undeformed (a) and deformed (b) state of a rectangular plate

The differential equation governing the longitudinal bending of the deformed plate $W(x, y)$ is given by:

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} + \frac{N\partial^2 W}{Db\partial x^2} = 0, \tag{1}$$

where $D = \frac{E\delta^3}{12(1-\mu^2)}$ is the bending rigidity; $\delta -$ plate thickness; $\mu -$ Poisson's ratio; $b -$ plate width; $L_n -$plate length; $L -$ distance between supports; $h_0 -$deflection at the center of the plate; $\rho_0 -$ radius of curvature at $x = \frac{L}{2}$; $E -$ elastic modulus; $0 \le x \le L_{n\,,} -\frac{b}{2} \le y \le \frac{b}{2}$. Boundary conditions:

$$x = 0, W = 0, \mathrm{Z}_1 = D\left(\frac{\partial^2 W}{\partial x^2} + \mu\frac{\partial^2 W}{\partial y^2}\right) = 0; \tag{2}$$

$$x = L_n, W = 0, \mathrm{Z}_1 = -D\left(\frac{\partial^2 W}{\partial x^2} + \mu\frac{\partial^2 W}{\partial y^2}\right) = 0. \tag{3}$$

Conditions at the free edges (where $H = D(1-\mu)\frac{\partial^2 W}{\partial x \partial y}$):

$$y = \mp\frac{b}{2} \text{ и } \begin{cases} \mathrm{Z}_2 = D(\frac{\partial^2 W}{\partial y^2} + \mu\frac{\partial^2 W}{\partial x^2}) = 0, \\ \frac{\partial \mathrm{Z}_2}{\partial y} + 2\frac{\partial^2 H}{\partial x} = 0, \end{cases} \tag{4}$$

The solution to equation (1) under conditions (2) is sought in the form

$$W = f(y) sin\frac{m\pi x}{L_n}. \tag{5}$$

Then for the function $f(y)$ we have the equation:

$$f_y^{(IV)} - 2\left(\frac{m\pi}{L_n}\right)^2 f_y'' + \left[\left(\frac{m\pi}{L_n}\right)^4 - \frac{N}{Db}\left(\frac{m\pi}{L_n}\right)^2\right] f(y) = 0, \tag{6}$$

under boundary conditions:

$$y = \mp\frac{b}{2}, \begin{cases} f_y'' - \mu\left(\frac{m\pi}{L_n}\right)^2 f(y) = 0, \\ f_y''' - (2-\mu)\left(\frac{m\pi}{L_n}\right)^2 f_y' = 0. \end{cases} \tag{7}$$

As shown in the work [4], a necessary condition for the existence of a solution (6) is the equality

$$f(y) = c_1 e^{\alpha y} + c_2 e^{-\alpha y} + c_3 e^{\beta y} + c_4 e^{-\beta y}, \tag{8}$$

$$\alpha = \sqrt{\sqrt{\frac{N}{Db}\left(\frac{m\pi}{L_n}\right)^2} + \left(\frac{m\pi}{L_n}\right)^2}, \beta = \sqrt{-\sqrt{\frac{N}{Db}\left(\frac{m\pi}{L_n}\right)^2} + \left(\frac{m\pi}{L_n}\right)^2}. \tag{9}$$

Then the solution to problem (6)-(7) can be written as

$$f(y) = 2\left[ch(\alpha y) - \frac{\alpha^2 - \mu\left(\frac{m\pi}{L_n}\right)^2}{\beta^2 - \mu\left(\frac{m\pi}{L_n}\right)^2} \cdot \frac{ch\frac{\alpha b}{2}}{ch\frac{\beta b}{2}} ch(\beta y)\right]. \tag{10}$$

Finally, for the deflection function *W* (*x*, *y*), we obtain the expression:

$$W(x,y) = 2\left[ch(\alpha y) - \frac{\alpha^2 - \mu\left(\frac{m\pi}{L_n}\right)^2}{\beta^2 - \mu\left(\frac{m\pi}{L_n}\right)^2} \cdot \frac{ch\frac{\alpha b}{2}}{ch\frac{\beta b}{2}} ch(\beta y)\right] sin\frac{m\pi x}{L_n}, \tag{11}$$

We set *m* = 1 and introduce the notation:

$$u = \frac{Nb}{D}, v = \left(\frac{\pi b}{L_n}\right)^2, \tag{12}$$

$$\alpha = \frac{\sqrt{v + \sqrt{v \cdot u}}}{b}, \beta = \frac{\sqrt{v - \sqrt{v \cdot u}}}{b}. \tag{13}$$

Since the value of v is always known, from this relation we can find the value of u. From (11), after replacing α and β with v, u, we can determine the deflection at any point in a segment of a flexible tape:

$$\left[\frac{(1-\mu)v + \sqrt{v \cdot u}}{(1-\mu)v - \sqrt{v \cdot u}}\right]^2 = \frac{\sqrt{v + \sqrt{v \cdot u}}}{\sqrt{v - \sqrt{v \cdot u}}} \cdot \frac{th\frac{\sqrt{v + \sqrt{v \cdot u}}}{2}}{th\frac{\sqrt{v - \sqrt{v \cdot u}}}{2}}, \tag{14}$$

$$W(x,y) = 2\left[ch\left(\frac{\sqrt{v + \sqrt{v \cdot u}}}{b} y\right) - \frac{(1-\mu)v + \sqrt{v \cdot u}}{(1-\mu)v - \sqrt{v \cdot u}} \times \right.$$

$$\left. \times \frac{ch\frac{\sqrt{v + \sqrt{v \cdot u}}}{2}}{ch\frac{\sqrt{v - \sqrt{v \cdot u}}}{2}} ch\left(\frac{\sqrt{v - \sqrt{v \cdot u}}}{b} y\right)\right] \mathrm{q} \cdot sin\frac{\pi x}{L_n}.$$

In the middle of the passage with $x = \frac{L_n}{2}, y = 0$, we find the quantity h_0:

$$h_0 = 2\left[1 - \frac{(1-\mu)v + \sqrt{v \cdot u}}{(1-\mu)v - \sqrt{v \cdot u}} \cdot \frac{ch\frac{\sqrt{v + \sqrt{v \cdot u}}}{2}}{ch\frac{\sqrt{v - \sqrt{v \cdot u}}}{2}}\right]. \tag{15}$$

The sought quantities ρ_0 and *N* are equal, respectively:

$$\rho_0 = \frac{b^2}{uh_0};\ N = \frac{uD}{b}. \tag{16}$$

The maximum value of the arrow of the transverse deflection is determined as the difference:

$$\Delta h_{max} = W\left(\frac{L_n}{2};\frac{b}{2}\right) - W\left(\frac{L_n}{2};0\right) = 2\left[\left(ch\frac{\sqrt{v+\sqrt{v\cdot u}}}{2} - 1\right) - \right.$$

$$\left. -\frac{(1-\mu)v+\sqrt{v\cdot u}}{(1-\mu)v-\sqrt{v\cdot u}}\cdot\frac{ch\frac{\sqrt{v+\sqrt{v\cdot u}}}{2}}{ch\frac{\sqrt{v-\sqrt{v\cdot u}}}{2}}\left(ch\frac{\sqrt{v-\sqrt{v\cdot u}}}{2} - 1\right).\right] \tag{17}$$

So, the whole difficulty of solving the formulated problem lies in finding u from equation (14). This equation for u can be solved by the method of successive approximations, taking for the zeroth approximation the value

$$u_0 = (1-\mu)v\left[1 + \frac{6\mu}{6-\mu v}\right].$$

Let us evaluate the effect of the load on the deformation of a rectangular plate. The geometric characteristics of a plate under the action of longitudinal compressive forces N (Fig.1) are reduced to two dimensionless parameters u and v. The quantity v is given, and u is determined from the transcendental equation (14). After simple transformations we get:

$$u = (1-\mu)v\left[1+\frac{6\mu}{6-\mu v}\right];\ 0 < v \le 0{,}2,$$
$$u = 1{,}0225v - 0{,}0263;\ 0{,}2 < v \le 1,$$
$$u = 0{,}99v;\ v > 1.$$

Considering that in practice the value of v is in the range $0 < v \le 0{,}2$, for the desired parameters h_0 and ρ_0 we obtain the expressions:

$$h_0^N = 2\left[1 + \frac{\sqrt{v\cdot u}+(1-\mu)v}{\sqrt{v\cdot u}-(1-\mu)v}\cdot\frac{8+v+\sqrt{v\cdot u}}{8+v-\sqrt{v\cdot u}}\right] \approx$$

$$\approx 3{,}34v + 13{,}34,\ \rho_0^N = \frac{b^2}{vh_0^N}.$$

Results. As an example of the application of the above methodology, in Table 1 compares the experimental data with the calculation results for the case L_n = 1270 mm, b =215 mm, μ = 0,3. The dependence of the maximum deflection h_0 on the parameter v is close to linear, as can be seen from Fig. 2.

Tab. 1. Comparison of results calculations with experiment

parameter	calculations	experiment
h_0	147,6	146,0
ρ_0	1252	1300
N	9,4 kG	10 kG
Δh_{max}	0,122 mm	0,125 mm

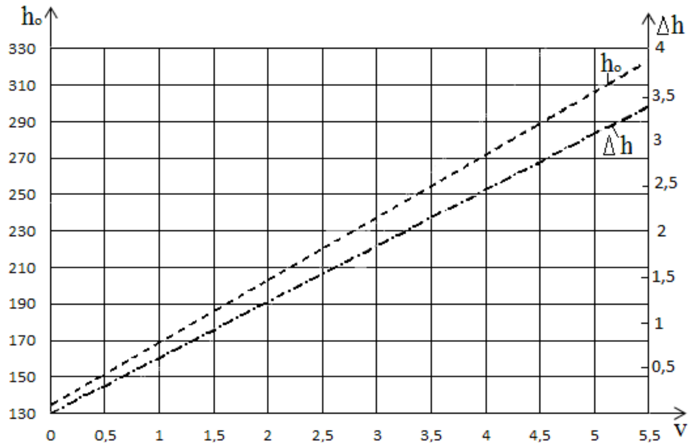


Fig. 2. The dependence of the transverse deflection on parameter v

Conclusion

Based on the data presented, it can be concluded that the accuracy of the calculations using the formulas obtained is sufficient for practice, subject to the permissible values of the parameters.

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