https://doi.org/10.26160/2474-5901-2020-18-26-29

THE CALCULATION OF THE SQUARE N-DIMENSIONAL SPHERE USING THE POISSON INTEGRAL

Tereshonok P.A., Borgoyakova Yu.L., Adamovich A.I.
Foreign language supervisor – ***Shelikhova S.V.***

Keywords: integral of the Euler-Pousson, n-sphere, volume of an n-dimensional ball.
Abstract. This paper examines the original method of calculating the volume of an n-dimensional ball, based on the definition of the body surface area in Minkowski.

ВЫЧИСЛЕНИЕ ПЛОЩАДИ N-МЕРНОЙ СФЕРЫ С ПОМОЩЬЮ ИНТЕГРАЛА ПУАССОНА

Терешонок П.А., Боргоякова Ю.Л., Адамович А.И.
Руководитель по иностранному языку – ***Шелихова С.В.***

Ключевые слова: интеграл Эйлера-Пуссона, n-сфера, объем n-мерного шара.
Аннотация. В настоящей заметке рассматривается оригинальный способ вычисления объема n-мерного шара, опираясь на определение площади поверхности тела по Минковскому.

A ball and an ellipse are necessary to describe any planetary system. For a student at an aerospace university, it is important to be able to obtain the volumes and areas of their multidimensional analogues. Based on the sources of [1-3], this paper proposes one of the options for solving this problem.

The set of points of an n-dimensional space whose coordinates satisfy the condition

$$G(a_1,...,a_n) := \left\{ x \in R^n : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots \frac{x_n^2}{a_n^2} \le 1 \right\}.$$

It called n-dimensional ellipsoid, for $a_1 = ... = a_n = a$ it goes into a ball of radius a. We will denote such a ball B_a, its n-dimensional volume $V_n(B_a)$, ball border, (n – 1) -dimensional sphere and its (n – 1) -dimensional “area”, respectively $\omega_a = \partial B_a$, $S_n(\omega_a)$. When a = 1, for convenience, we put $V_n = V_n(B_1)$, $S_n = S_n(\omega_1)$.

Variable Replacement $x_j = ay_j;\, j = 1,...,n$, has a jacobian transition $I = a^n$, consequently,

$$V_n(B_a) = \iint\limits_{B_a} ... \int dx_1...dx_n = \iint\limits_{B_1} ... \int a^n dy_1...dy_n = a^n V_n(B_1).$$

Similarly, changing variables $x_j = a_j y_j;\, a_j > 0;\, j = 1,...,n$, has a jacobian transition $I = a_1...a_n$ and converts an n-dimensional ellipsoid $G(a_1,...,a_n)$ into a unit ball, therefore $V_n(G(a_1,...,a_n)) = a_1...a_n V_n$. Calculating the constant V_n, we get formulas for the volume of an n-dimensional ball and an ellipsoid.

Because we already know the dependence of the volume of the ball on the radius, then

$$V_n = \underset{x_1^2+\ldots x_n^2\le 1}{\iint\ldots\int} dx_1\ldots dx_n = \int_{-1}^{1} dx_n \underset{x_1^2+\ldots x_{n-1}^2\le 1-x_n^2}{\iint\ldots\int} dx_1\ldots dx_{n-1} = V_{n-1}\int_{-1}^{1}\left(1-x_n^2\right)^{\frac{n-1}{2}} dx_n .$$

We indicate the relationship between the n-dimensional volume of the ball $V_n(B_a)$ and (n – 1)- dimensional area of an (n–1)-dimensional sphere $S_n(\omega_a)$. From geometric considerations, it is clear that, by analogy with the volume of an n-dimensional ball, for (n–1)-dimensional sphere $S_n(\omega_a)=a^{n-1}S_n(\omega_1)=a^{n-1}S_n$. From the definition of a derivative $\left(V_n(B_a)\right)'=S_n(\omega_a)$, here the derivative is taken along the length of the radius a.

Therefore

$$S_n(\omega_a)=\frac{dV_n(B_a)}{da}=\frac{da^nV_n}{da}=na^{n-1}V_n .$$

and therefore , $nV_n=S_n$ – relationship between the volume and area of a single n-ball.

The well-known Euler-Poisson integral

$$I=\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi} .$$

We calculate the n-degree of the Euler-Poisson integral, first replacing it with a multiple integral over the entire space R^n, and then we reduce it to a repeated integral, and take the first integral over the (n – 1) -dimensional sphere of radius r, and then integrate over r from 0 to ∞.

$$\left(\sqrt{\pi}\right)^n = I^n = \left(\int_R e^{-x^2}dx\right)^n = \int_{R^n} e^{-x_1^2-\ldots-x_n^2}dx = \int_0^{\infty} e^{-r^2}dr \int_{x_1^2+\ldots+x_n^2=r^2} ds ,$$

$$\int_{x_1^2+\ldots+x_n^2=r^2} ds = S_n(r)=r^{n-1}S_n(\omega_1)=r^{n-1}S_n ,$$

then we obtain the following equality.

$$\left(\sqrt{\pi}\right)^n=\int_0^{\infty}e^{-r^2}r^{n-1}S_n dr=S_n\int_0^{\infty}e^{-r^2}r^{n-1}dr$$

The integral, in the last equality, is reduced to a gamma function by means of a replacement.

$$\int_0^{\infty}e^{-r^2}r^{n-1}dr=\left|r^2=t, dt=2rdr\right|=\int_0^{\infty}e^{-t}t^{\frac{n-1}{2}}\frac{1}{2}t^{-\frac{1}{2}}dt=\frac{1}{2}\int_0^{\infty}e^{-t}t^{\frac{n}{2}-1}dt=\frac{1}{2}\Gamma\left(\frac{n}{2}\right).$$

We obtain the equation for S_n, solving it, we find $S_n=2\pi^{\frac{n}{2}}\left(\Gamma\left(\frac{n}{2}\right)\right)^{-1}$,

respectively $S_n(\omega_a)=S_na^{n-1}=2\pi^{\frac{n}{2}}a^{n-1}\left(\Gamma\left(\frac{n}{2}\right)\right)^{-1}$.

Since $nV_n = S_n$, then

$$V_n = \frac{S_n}{n} = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)};\ V_n\left(B_a\right) = \frac{\pi^{\frac{n}{2}}a^n}{\Gamma\left(\frac{n}{2}+1\right)};V_n(G(a_1,...,a_n)) = \frac{a_1...a_n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Note that the formula for the surface area of the ellipsoid is absent here.

However, back to the volume of the n-dimensional ball. For the cases of even and odd n, we obtain the formulas

$$V_{2n} = \frac{\pi^n}{n!};\quad V_{2n+1} = \frac{2(2\pi)^n}{(2n+1)!}.$$

The first three meanings are well known. $V_1 = 2$, $V_2 = \pi \approx 3{,}141593$, $V_3 = \frac{4\pi}{3} \approx 4{,}18879$.

We write four more values $V_4 = \frac{\pi^2}{2} \approx$ 4,934802, $V_5 = \frac{8\pi^2}{15} \approx$ 5,263789, $V_6 = \frac{\pi^3}{6} \approx$ 5,167713, $V_7 = \frac{16\pi^3}{105} \approx$ 4,724766.

Note that for the dimension of space we can write down the formula $n = [V_n] - 1$, which is true for the first three values, for the next two values there is equality $n = [V_n]$. We will increase the dimension. The volume of a single cube does not change and remains equal to unity. And the volume of an n-dimensional ball, with an unlimited increase in dimension, tends to zero. This follows from the fact that factorial grows faster than any power function. So, if we consider the volume of an n-dimensional ball as a function of the dimension of space, then as n increases, this function tends to zero. This function has only one extremum n = 5.

The probabilistic meaning of this fact is as follows. If a point is randomly thrown into a single n-dimensional cube containing a part of an n-dimensional ball lying in the first orthant, then the probability that we get into the ball tends to zero with increasing dimension.

Reference

1. Zorich V.A. Sphere, sphere and all-all-all // Bulletin of Moscow University. Series 1. Mathematics. Mechanics. 2016. No. 3. P. 16-19.
2. Zaslavsky A.A. On the calculation of the volume of an n-dimensional ball // Mathematical education. Ser. 3. 2008. No. 12. P. 270-271.
3. Neklyudova A.V. Some non-standard proofs and problems in the course of mathematical analysis // Engineering Journal: Science and Innovation. 2013. No. 5. Electronic resource, URL http://engjournal.ru/catalog/pedagogika /hidden/743.html.

Список литературы

1. Зорич В.А. Шар, сфера и всё- всё- всё // Вестник Московского университета. Серия 1 .Математика. Механика. 2016. №3. С. 16-19.
2. Заславский А.А. О вычислении объема n-мерного шара // Математическое просвещение. Сер. 3. 2008. Вып. 12. С. 270-271.
3. Неклюдова А.В. Некоторые нестандартные доказательства и задачи в курсе математического анализа // Инженерный журнал: наука и инновации. 2013. Вып. 5. Электронный ресурс, URL http://engjournal.ru/ catalog/pedagogika/hidden/743.html.

Терешонок Павел Александрович – студент	**Tereshonok Pavel Aleksandrovich** – student
Боргоякова Юлия Львовна – студент	**Borgoyakova Yulia L'vovna** – student
Адамович Анатолий Игоревич – студент, anatoliyadamovich@icloud.com	**Adamovich Anatoliy Igorevich** – student, anatoliyadamovich@icloud.com
Шелихова Светлана Викторовна – старший преподаватель	**Shelikhova Svetlana Viktorovna –** senior lecturer
Сибирский государственный университет науки и технологий имени академика М.Ф. Решетнёва, г. Красноярск, Россия	Siberian state university of science and technology named after academician M.F. Reshetnyov, Krasnoyarsk, Russia

Received 05.02.2020