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REDUCED DYNAMIC STIFFNESS IN STRUCTURAL MODELS OF TECHNOLOGICAL AND TRANSPORT MACHINES

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Keywords: dynamic rigidity, quasi-spring, dynamic reaction, form of motion.

Abstract. The article focuses on the problems of constructing mathematical models of mechanical oscillatory systems containing elastic and mass-and-inertia elements. It shows the possibilities of combining typical elementary links in some structural formations, called quasi-springs. Such constituent links have the same properties as conventional springs in commutation and transformation circuits. Features of dynamic stiffness of system in general and quasi-springs, in particular, are that dynamic stiffnesses take zero values and infinitely large ones. The authors propose a method for determining the frequencies of natural oscillations on the basis of using the concepts of the dynamic stiffness of a system. The work considers features of the formation of dynamic responses in systems whose elements make joint motions.

ПРИВЕДЕННАЯ ДИНАМИЧЕСКАЯ ЖЕСТКОСТЬ В СТРУКТУРНЫХ МОДЕЛЯХ ТЕХНОЛОГИЧЕСКИХ И ТРАНСПОРТНЫХ МАШИН

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Ключевые слова: динамическая жесткость, квазипружина, динамическая реакция, форма движения

Аннотация. Рассматриваются вопросы построения математических моделей механических колебательных систем, содержащих в своем составе упругие и массоинерционные элементы. Показаны возможности соединения типовых элементарных звеньев в некоторые структурные образования, называемые квазипружинами. Такие составные звенья обладают в схемах коммутации и преобразования такими же свойствами, что и обычные пружины. Особенности динамической жесткости системы в целом и квазипружин, в частности, заключаются в том, что динамические жесткости принимают нулевые значения, а также – бесконечно большие. Предложен метод определения частот собственных колебаний на основе использования понятий о динамической жесткости системы. Рассмотрены особенности формирования динамических реакций в системах, элементы которых совершают совместные движения.

Introduction. Manifestations of joint motions on several coordinates in the forms arising at free oscillations of systems, the modes of dynamic damping of oscillations and also in the joint motions along several coordinates with observance of certain lever ratios are characteristic of mechanical oscillatory systems with several degrees of freedom [1–4]. Many manifestations of self-organization of motions are considered in works on synergetics, nonlinear mechanics, the theory of gyroscopic systems and dynamics of the coupled systems of solid bodies. Physical bases of such processes are determined by impact of various factors, are followed by difficult interactions of various components of elements that makes the directions of search of ways and means of mathematical modeling relevant [5–9].

In most cases the small oscillations are considered that arise in mechanical oscillatory systems in relation to the position of the established dynamic state at limited number of degrees of freedom of motion, which predetermines the advisability of structural mathematical methods; it found reflection in works [8, 10, 11]. At the same time, questions of formation of joint motions along several coordinates and conditions of their implementation demand more detailed representations.

The offered article considers the problems of self-organization of the motions of elements of mechanical oscillatory systems at external harmonic perturbations taking into account opportunities of manifestation of lever linkages between the elements of the system and features of the dynamic stiffnesses which are formed in interactions at certain forms of motion.

I. General provisions. Statement of the research task. The definition of the concept of dynamic stiffness in mechanical oscillatory systems (or computational schemes of technical objects) is associated with the expansion of ideas about the elastic and mass-and-inertia interactions of structural elements (quasi-springs) in the composition of mechanical systems. Some questions about the use of dynamic stiffnesses in the form of quasi-springs are described in [12].

Further on, the problems of vibration protection are considered, which is connected with the allocation of the protection object and the system of constraints, the adjustment of which ensures the corresponding dynamic states. In the problems of the dynamics of machines of this kind, these representations are completely compatible, since the presence of an object, whose dynamic state is evaluated, is characteristic of many applications.

Fig. 1, *a* – *c* shows computational and structural schemes of vibration protection system with two degrees of freedom in which the object of protection with the mass m_1, the intermediate mass m_1 are allocated. Mass-and-inertia elements m_1 and m_2 connect linear springs to coefficients of stiffness k_1, k_2 and k_3. External perturbations are accepted in the form of harmonic forces *of* Q_1 and Q_2, applied to elements m_1 and m_2; kinematic perturbations z_1 and z_2 are defined by the motion of bearing surfaces *I* and *II* (Fig. 1, *a*).

The motion of the system is considered in coordinates y_1 and y_2 connected with the motionless basis. Within a structural approach, the starting point in the research is the construction of a mathematical model in the form of the system from two linear differential equations of the second order with constant coefficients. That is further compared to the dynamically equivalent structural diagram of the automatic control system, in which the object of protection is considered as the integrating link of the second order [3, 5, 6]. The structural diagram of the system consists of two partial blocks with an elastic interpartial constraint (Fig. 1). Construction of the structural diagram (Fig. 1, *b*) is based on preliminary Laplace transformation at zero initial conditions. At the same time in the structural diagram in Fig. 1, reflecting the existence of negative feedbacks (quasi-springs) which are formed in characteristic points (pp. A_1, A_2, A_3 and B_1, B_2, B_3).

The dynamic constraint response which is created with relation to the object of protection, is defined as the product of the corresponding dynamic displacement $\bar{y}_1$ by dynamic stiffness of the quasi-spring. The complete response consists of static and dynamic parts. Static components are determined by the weight forces of the object of protection and the intermediate element.

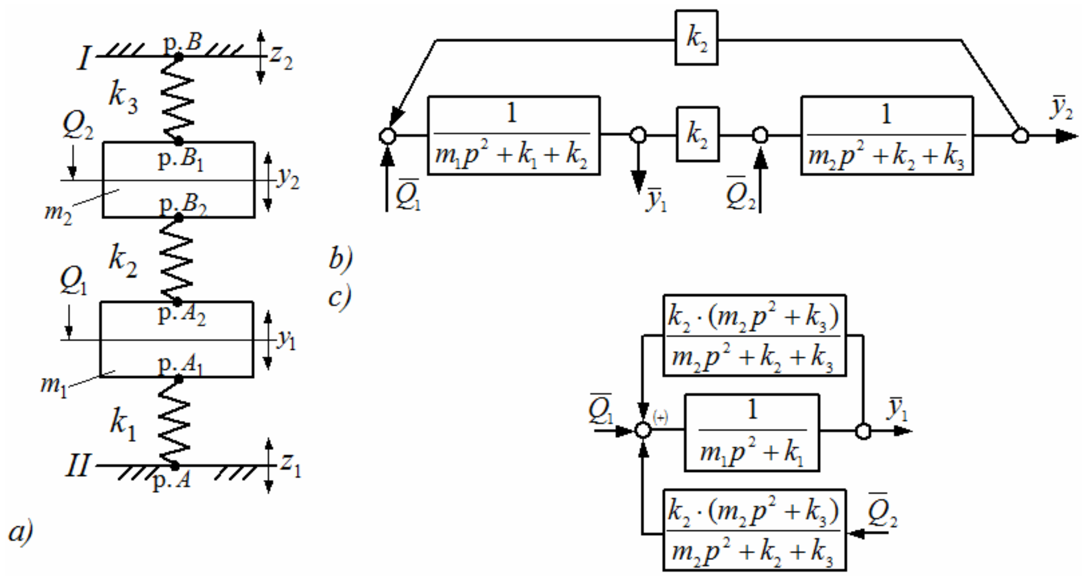

Fig. 1. Computational schemes and structural diagrams of the vibration protection system: *a* is the chain-type oscillating system*; b* is the detailed structural diagram*; c* is the generalized structural diagram

II. Formation of dynamic stiffness. Transfer function of vibration protection system at force perturbation *of* $Q_1 \neq 0$ ($Q_2 = 0$, $z_1 = 0$, $z_2 = 0$) is defined from the structural diagram in Fig. 1*:*

$$W_1(p) = \frac{\bar{y}_1}{\bar{Q}_1} = \frac{m_2p^2 + k_2 + k_3}{A(p)}; \quad (1)$$

where

$$A(p) = (m_1p^2 + k_1 + k_2)\cdot(m_2p^2 + k_2 + k_3) - k_2^2 \qquad (2)$$

is the characteristic frequency equation of the system.

In this case $p = j\omega$ is a complex variable ($j = \sqrt{-1}$); the sign "–" means the Laplace transform image of the variable [5]. The physical sense of transfer function in the form of (1) corresponds to dynamic compliance ($\bar{\Pi}$), which is defined by the expression

$$\bar{\Pi} = \frac{\bar{y}_1}{\bar{Q}_1} = \frac{1}{m_1p^2 + k_1 + \dfrac{k_2\cdot(m_2p^2 + k_3)}{m_2p^2 + k_2 + k_3}}. \qquad (3)$$

As for dynamic stiffness of system, it corresponds to inversion of expression (3):

$$\bar{k}_{\text{пр}}(p) = \frac{\bar{Q}_1}{\bar{y}_1} = \frac{(m_1p^2 + k_1)\cdot(m_2p^2 + k_2 + k_3) + k_2\cdot(m_2p^2 + k_3)}{m_2p^2 + k_2 + k_3}; \quad (4)$$

$$\bar{k}_{\text{пр}}(p) = (m_1 p^2 + k_1) + \frac{k_2 \cdot (m_2 p^2 + k_3)}{m_2 p^2 + k_2 + k_3} = m_1 p^2 + \frac{k_1 \cdot (m_2 p^2 + k_2 + k_3) + k_2 \cdot (m_2 p^2 + k_3)}{m_2 p^2 + k_2 + k_3} =$$
$$= m_1 p^2 + \frac{m_2 p^2 \cdot (k_1 + k_2) + k_1 k_2 + k_1 k_3 + k_2 k_3}{m_2 p^2 + k_2 + k_3}. \quad (5)$$

The dynamic stiffness of structural formations (quasi-springs) is determined by fractional-rational expressions, which in the general case depend on the frequency of the external force. Note that dynamic stiffness, by definition, can take zero, and also negative and positive values, including those that reach large values.

Thus, the methodical basis of an estimation of dynamic properties of complex systems relies upon the possibilities of structural transformations of the initial mathematical model.

III. Features of a system with one degree of freedom. A mechanical oscillatory system with one degree of freedom with power perturbation $\overline{Q}$ and kinematic perturbation $\overline{z}$ from a bearing surface *I* is shown in Fig. 2, *a*. As a part of the system, there is an object of protection from vibration by the mass *m*, an elastic element with the stiffness *k* and the motion transformation device (MTD) with the reduced mass *L*. Such systems are usually used as computational schemes in the problems of protecting machines and equipment from vibrations and are called basic ones; more complex systems, at an elimination of intermediate coordinates, can be reduced to them [3, 5, 9].

The transfer functions of the system for the force and kinematic perturbations, respectively, have the form

$$W_1(p) = \frac{\bar{y}}{\overline{Q}} = \frac{1}{(m+L)p^2 + k}; \ (6) \quad W_2(p) = \frac{\bar{y}_1}{\bar{z}} = \frac{Lp^2 + k}{(m+L)p^2 + k}. \ (7)$$

Fig.2, *c* shows the transformed structural diagram, in which the protection object is allocated in the form of an integrating link of the second order, enclosed by a negative feedback loop. Such a loop, in itself, has a transfer function of the form

$$\underset{OC}{W}(p) = Lp^2 + k. \quad (8)$$

In physical sense, the expression (8) defines the dynamic stiffness of the elastic and inertial system (or the vibration isolator) (Fig. 2, *a*) between the object of protection *m* and the bearing surface *I*. In this case, the expression $Lp^2 + k$, being the dynamic stiffness of some structural formation, also defines dynamic responses at pp. *A* and A_1 , where the elastic formation (quasi-spring) is connected to the bearing surface *I* and the object of protection *m*.

Responses at pp. *A* and A_1 are defined at force perturbation $\overline{Q}$ ($z = 0$) expression

$$\left|\overline{R}_A\right| = \left|\overline{R}_{A_1}\right| = (Lp^2 + k) \cdot \bar{y} = \frac{\overline{Q} \cdot (Lp^2 + k)}{(m+L)p^2 + k}. \quad (9)$$

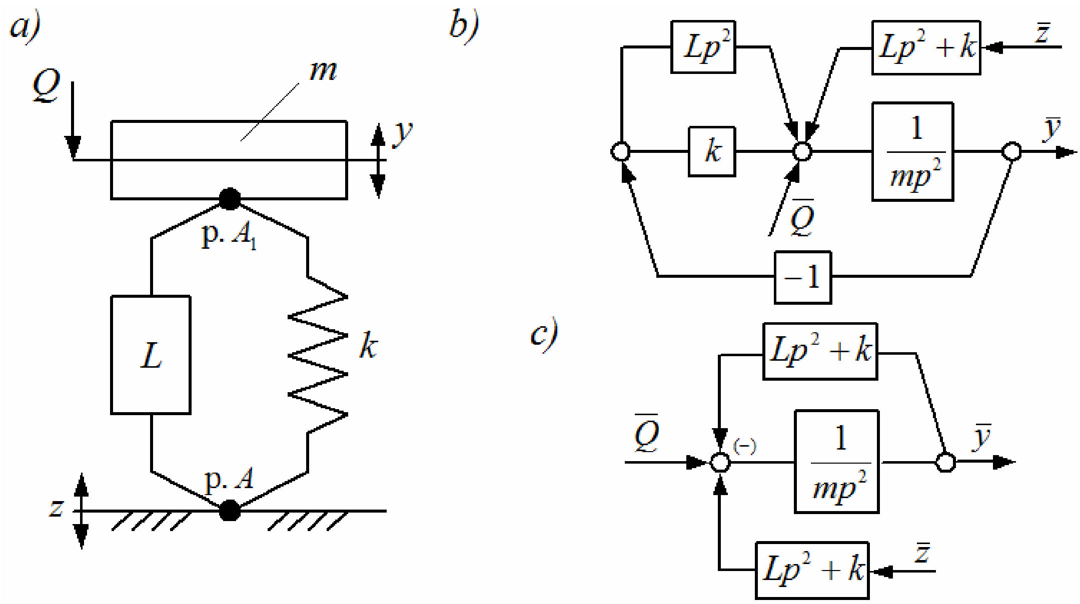

Fig. 2. Computational scheme and structural diagram of the vibration protection system with one degree of freedom: *a* is the computational scheme; *b* is the structural diagram at the kinematic perturbation ($\bar{Q}=0$); *c* is the structural diagram of the system at the force perturbation ($\bar{z}=0$)

The transfer function of the system at the initial influence $\bar{Q}$ and an output signal in the form of dynamic response $\bar{R}_A$ or $\bar{R}_{A_1}$ can be written

$$W_{\substack{R_A \\ \bar{z}=0}}(p)=\frac{\bar{R}_A}{\bar{Q}}=\frac{Lp^2+k}{(m+L)p^2+k}. \tag{10}$$

The significance of dynamic responses, occurring in the connections of system elements, allows us to expand methodological positions in solving problems of ensuring the reliability of the operation of vibration protection systems.

IV. Some features of dynamic stiffness. Using the transfer function (5), it is possible to find the dynamic stiffness of a fragment of the system (Fig. 2, *a*)

$$\bar{k}_{\text{пр}}(p)=\frac{\bar{Q}}{\bar{y}}=(m+L)p^2+k. \tag{11}$$

Dynamic stiffness of a fragment of system $\bar{k}_{\text{пр}}(p)$ has two components: one is a constant with a value k and corresponds to the stiffness of the elastic element; the second one is a variable and depends on the frequency of the external disturbance $(m + L)\omega^2$. The dynamic stiffness of a fragment of the system (vibration isolator) can take a zero value, which occurs at a frequency

$$\omega^2=\frac{k}{m+L}. \tag{12}$$

Such frequency defines natural or free oscillations of the system, in general, at the same time the denominator of transfer function (8) is the frequency characteristic equation. Fig.3 shows a graphic-analytical solution of the frequency equation

$$-(m+L)\omega^2 = 0 \tag{13}$$

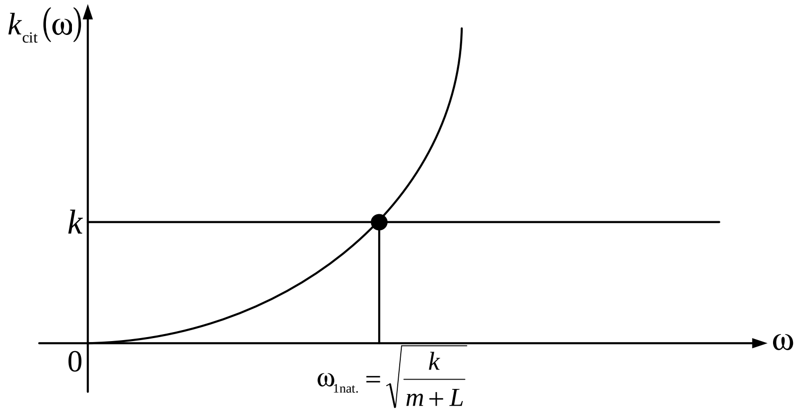

Fig. 3. A schematic diagram of the graphic-analytical solution of equation (13)

In this case the dynamic stiffness of the system at force perturbation $\overline{Q}$ consists of two parts: the first one considers inertial properties of the object of protection *m*, and the second one considers the dynamic properties of the compound elastic element obtained at parallel connection of a spring *k* and MTD with the reduced mass *L*.

1. If the dynamic stiffness of the system is zero, a motion with an infinitely large displacement amplitude (the system compliance or mass compliance is infinitely large) is implemented, that is, a mode of resonant phenomena occurs.

2. If dynamic stiffness of an elastic component (or quasi-springs) is equal to zero, then the motion of the object of protection *m* will not be resonant – the object of protection will make harmonic oscillations with a frequency $\omega_0^2 = \dfrac{k}{L}$ and the corresponding amplitude.

3. For the kinematic influence *z* from the side of the bearing surface (Q = 0), the transfer function of the system has the form determined by the expression (7). At the same time the expression (7) gives an idea not of dynamic stiffness of system, per se, but characterizes the relation of amplitudes of oscillations $\bar{y}/\bar{z}$ at the object of protection and the bearing surface *I*. In this case, the analysis of dynamic interactions is considered from a different point of view. If to give kinematic perturbation $(Lp^2 + k)\ \bar{z}$ to equivalent power influence

$$\overline{Q}_{\text{экв.}}(p) = \bar{z}\cdot(Lp^2 + k), \tag{14}$$

that transfer function $\bar{y}/\overline{Q}_{\text{экв.}}$ the same form, as well as expression will take a form (7). Accounting of the specifics of kinematic perturbation as a relation of $\bar{y}/\bar{z}$ has the features owing to manifestation of the so-called lever linkages arising at the joint motions of elements in oscillatory systems.

As for the system with two degrees of freedom, then the interaction of elements in the system, taking into account the forming dynamic stiffnesses of the system, as a whole, and its individual elastic structures or formations, is more complex.

V. Dynamic stiffness of the system. As was shown above, the generalized dynamic stiffness can be obtained by inversion of the expression for dynamic compliance (or the transfer function of the "input" - "force", "output" - "displacement" system). The generalized dynamic stiffness reflects the level of dynamic interactions on the part of the system (in the form of resistance) at the point of application of the harmonic force. The generalized dynamic stiffness of the system is defined by the expression

$$\bar{k}_{\text{об.}}(p) = \frac{\bar{Q}}{\bar{y}} = \frac{(m_1 p^2 + k_1 + k_2)\cdot(m_2 p^2 + k_2 + k_3) - k_2^2}{m_2 p^2 + k_2 + k_3}. \tag{15}$$

From (15) it follows that at a frequency of dynamic damping of oscillations (when the "object of protection" stops)

$$\underset{\omega_1 = 0}{\omega^2_{\text{дин.}}} = \frac{k_2 + k_3}{m_2}, \tag{16}$$

the generalized dynamic stiffness of the system will be equal to ∞.

At the same time, at frequencies ω_1 and ω_2 of natural oscillations, the generalized stiffness of the system will be equal to 0, thereby eliminating the resistance for the motion of an object with the mass m_1. On the other hand, the computational scheme (Fig. 4, *a*) can be transformed to a form in which the initial system takes a form of a system with one degree of freedom. In a system like that, the quasi-spring can be implemented as some structural formation made of elastic and mass-and-inertia elements. Similar transformations can be carried out in the structural diagram (Fig. 4, *c*).

The quasi-spring represents a certain block which transfer function is formed according to the certain algorithm, built using the rules of structural transformations of the theory of mechanical circuits. Stages of development of ideas of the quasi-spring found their reflection in work [6]. The structural diagram in Fig. 4 has in relation to an object of protection $\left(\frac{1}{m_1 p^2}\right)$ negative feedback.

The transfer function of such a feedback, in the physical sense, determines the dynamic stiffness of the quasi-spring, which corresponds to the usual structural diagrams in systems with one degree of freedom. The special properties of the quasi-spring are in the fact that at certain frequencies of external action it can take extreme values. So, for example, with a frequency $\omega_{\text{дин.}} = \sqrt{\frac{k_2 + k_3}{m_2}}$ dynamic stiffness of a quasi-spring, in versions of structural diagrams in Fig. 4, *c* and Fig. 4, becomes equal to ∞, and coordinate $\bar{y}_1$ takes zero value. At the same time the generalized dynamic stiffness determined by the expression (15) also becomes equal to ∞. Such forms of interaction are quite explainable at a stop of the motion along the coordinate $\bar{y}_1$. In the structural diagrams provided in Fig. 4, *c* and Fig. 4, in such cases the negative feedback becomes equal to ∞ with the corresponding "zeroing" of the coordinate $\bar{y}_1$.

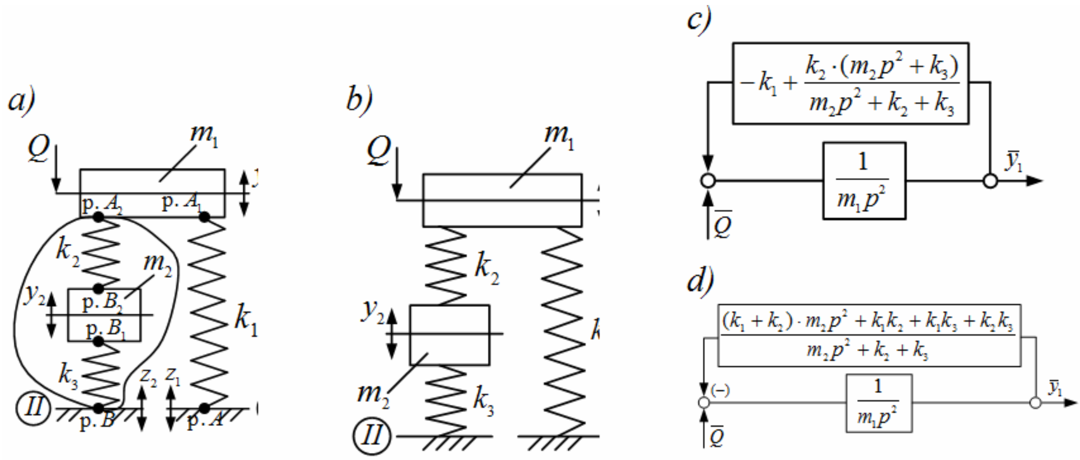

Fig. 4. Computational schemes and structural diagrams of the system: *a* is the object is rested against a quasi-spring and a spring k_1 having autonomous external influences; *b* is the object is based on a generalized quasi-spring with a common support; *c* is the structural diagram corresponding to Fig.4, *a; d* is the structural diagram corresponding to Fig.4, *b*

Conclusion

1. The concept of dynamic stiffness in mechanical oscillation systems is correlated with representations about the properties of the system as a whole, and also with the properties of individual fragments of the system.

2. The dynamic stiffness of the system as a whole is determined on the basis of the transfer function, which connects the input disturbance and the output displacement of the point to which the external disturbance is applied. Such dynamic stiffness depends on the frequency of the external action and can be represented by a fractional-rational expression in the operator form. If the dynamic stiffness of the system as a whole is zero, this corresponds to the resonance regime. Using the fact that the dynamic stiffness of the system, in the physical sense, is determined by the frequency characteristic equation, the frequencies of the natural oscillations can be determined from the condition that the dynamic stiffness of the system is equal to zero on the basis of graphic-analytical methods.

3. Dynamic stiffness can be correlated with the properties of individual fragments. Among them, quasi-springs can be classified as some structural formations built on connections of typical elementary links according to the rules of the structural theory of mechanical oscillatory systems. Within the scope of such theory, a mechanical oscillatory system with lumped parameters, which has a mathematical model in the form of a system of linear ordinary differential equations, can be associated with a structural mathematical model in the form of a structural diagram of a dynamically equivalent automatic control system.

Список литературы

1. Лилов Л.К. Моделирование систем связанных тел. – М.: Наука, 1993. – 272 с.
2. Хакен Г. Синергетика. – М.: Мир, 1980. – 404 с.
3. Елисеев С.В. Динамическое гашение колебаний: концепция обратной связи и структурные методы математического моделирования / С.В. Елисеев, А.П. Хоменко. – Новосибирск: Наука, 2014. – 357 с.
4. Коловский М.З. Автоматическое управление виброзащитными системами. – М.: Наука, 1976. 320 с.
5. Хоменко А.П. Метод структурных преобразований и его приложения в задачах динамики виброзащитных систем. Определение реакций связи / А.П. Хоменко, С.В. Елисеев, Р.С. Большаков // Современные технологии. Системный анализ. Моделирование. – 2016. – №1(41). – С. 8-23.
6. Елисеев С.В. Прикладная теория колебаний в задачах динамики линейных механических систем / С.В. Елисеев, А.И. Артюнин. – Новосибирск: Наука, 2016. – 459 с.
7. Ким Д.П. Теория автоматического управления. Т. 1. Линейные системы. – М.: ФИЗМАТЛИТ, 2003. 288 с.
8. Karnovsky I.A. Theory of vibration protection / I.A. Karnovsky, E. Lebed. – Switzerland: Springer, 2016. – 708 p.
9. Редько С.Ф. Идентификация механических систем. Определение динамических характеристик и параметров / С.Ф. Редько, Ушкалов В.Ф., В.П. Яковлев. – Киев: Наук.думка, 1985. – 216 с.
10. Тарасик В.П. Математическое моделирование технических систем / В.П. Тарасик. 2-е изд. испр. и доп. – Минск: Дизайн ПРО, 2004. – 640 с.
11. Clarence W. Vibration. Fundamentals and Practice / Clarence W., De Silva. – Boca Raton, London, New York, Washington, D.C.: CRC Press, 2000. – 957 p.
12. Хоменко А.П. Квазиэлементы в механических колебательных системах. Особенности систем при исключении переменных динамического состояния / А.П. Хоменко, С.В. Елисеев // Современные технологии. Системный анализ. Моделирование. – № 2(38). – 2013. – С. 8-17.

References

1. Lilov L.K. Modeling of systems of coupled bodies. Moscow: Nauka Publ., 1993. 272 p.
2. Haken G. Synergetics. Moscow: Mir Publ., 1980. 404 p.
3. Eliseev S.V., Khomenko A.P. Dynamic damping of oscillations: the concept of feedback and structural methods of mathematical modeling. Novosibirsk: Nauka Publ., 2014. 357 p.
4. Kolovsky M.Z. Automatic control of vibration protection systems. - Moscow: Nauka, 1976. 320 p.
5. Khomenko A.P., Eliseev S.V., Bolshakov R.S. The method of structural transformations and its applications in the problems of the dynamics of

vibration protection systems. Determination of constraint responses. Modern technologies. System analysis. Modeling, 2016. No. 1 (41). Pp. 8-23.

6. Eliseev S.V., Artyunin A.I. Applied theory of oscillations in problems of the dynamics of linear mechanical systems. Novosibirsk: Nauka Publ., 2016. 459 p.
7. Kim D.P. Theory of automatic control. Vol. 1. Linear systems. Moscow: FIZMATLIT Publ., 2003. 288 p.
8. Karnovsky I.A., Lebed. E. Theory of vibration protection. Switzerland: Springer, 2016. 708 p.
9. Red’ko S.F., Ushkalov V.F., Yakovlev V.P. Identification of mechanical systems. Determination of dynamic characteristics and parameters.. Kiev: Nauk.dumka Publ., 1985. 216 p.
10. Tarasik V.P. Mathematical modeling of technical systems. 2nd ed. Revised and enlarged. Minsk: Design of missile defense, 2004. 640 p.
11. de Silva C.W. Vibration. Fundamentals and Practice. Boca Raton, London, New York, Washington, DC: CRC Press, 2000. 957 p.
12. Khomenko A.P., Eliseev S.V. Quasi-elements in mechanical oscillatory systems. Features of systems with the exclusion of variables of the dynamic state. Modern technologies. System analysis. Modeling. No. 2 (38). 2013. Pp. 8-17.

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